Lecture 8

This is a review class. We will provide solutions for the practice problems.

We start with problem #5, #6, #7.

Recall

Thm: Consider the D.E:

$$ay'' + by' + cy = 0 \qquad (1)$$

Its characteristic eqn

$$a\lambda^2 + b\lambda + c = 0 \qquad (2)$$

Case (I): If (2) has two distinct real roots, $\Delta = b^2 - 4ac$

$\lambda_1 \neq \lambda_2$ λ_1, λ_2 $(\Delta > 0)$, then

the general solns of (1) are

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Case (II): If (2) has only one repeated real root λ_0 $(\Delta = 0)$, then the general solns of (1) are,

$$y = c_1 e^{\lambda_0 x} + c_2 x e^{\lambda_0 x}$$

Case (III): If (2) has two distinct complex roots $\alpha + i\beta$, $\alpha - i\beta$ $(\Delta < 0)$, then the general solns of (1) are

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

#5. Solve I.V.P

$$y'' - 2y' - 3y = 0 \qquad \text{D.E}$$

$$y(0) = 3,\ y'(0) = 3 \qquad \text{initial condition}$$

A: Step 1: Find the general solns of D.E

The characteristic eqn

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\Rightarrow$ Two distinct real roots: $\lambda_1 = 3$, $\lambda_2 = -1$

$\Rightarrow$ The general solns of (1) are

$$y(x) = C_1 e^{3x} + C_2 e^{-x}$$

Step 2: Use the initial conditions

$y(0) = 3 \Rightarrow$ let $x=0$, $y=3$

$$3 = C_1 e^0 + C_2 e^0 \Rightarrow C_1 + C_2 = 3 \quad ①$$

Note

$$y'(x) = 3C_1 e^{3x} - C_2 e^{-x}$$

$y'(0)=3 \Rightarrow$

$$3 = 3c_1 e^0 - c_2 e^0 \Rightarrow 3c_1 - c_2 = 3 \quad ②$$

Combining ①, ② $\Rightarrow$

$$\begin{cases} c_1 + c_2 = 3 \\ 3c_1 - c_2 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{6}{4} = \frac{3}{2} \\ c_2 = \frac{3}{2} \end{cases}$$

Hence the soln to the I.V.P is

$$y(x) = \frac{3}{2} e^{3x} + \frac{3}{2} e^{-x}$$

#6. Solve I.V.P

$$y'' + 4y' + 4y = 0$$

$$y(0) = 1, \quad y'(0) = 3$$

A: Step1: Find the general soln of D.E

The characteristic eqn

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow \quad (\lambda + 2)^2 = 0$$

Thus it has only one repeated root $\lambda = -2$.

Thus the general solns of the D.E are

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

Step2: Use the initial conditions:

$x=0$
$y=1$

$y(0)=1 \Rightarrow 1 = c_1 + c_2 \cdot 0 \Rightarrow c_1 = 1$

Note

$$y'(x) = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$y'(0)=3 \Rightarrow 3 = -2c_1 + c_2 \Rightarrow c_2 = 3 + 2c_1 = 5$

$x=0$
$y'=3$

Hence the soln to the I.V.P is

$$y = e^{-2x} + 5xe^{-2x}$$

#7. Solve the I.V.P

$$y'' - y' + y = 0 \qquad \text{D.E}$$

$$y(0)=1, \; y'(0)=3. \qquad \text{initial conditions}$$

A: Step 1: Find the general solns of D.E

The char. eqn

$$\lambda^2 - \lambda + 1 = 0$$

$a=1$
$b=-1$
$c=1$

$$\Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

$$= \frac{1 \pm \sqrt{3}\, i}{2}$$

Hence two complex roots:

$$\lambda_1 = \underbrace{\tfrac{1}{2}}_{\alpha} \underbrace{+}_{+} \underbrace{\tfrac{\sqrt{3}}{2}\, i}_{\beta i}, \quad \lambda_2 = \underbrace{\tfrac{1}{2}}_{\alpha} \underbrace{-}_{-} \underbrace{\tfrac{\sqrt{3}}{2}\, i}_{\beta i}$$

$$\Rightarrow \quad \alpha = \frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$$

Hence the general solns of the D.E are

$$y(x) = c_1 e^{\frac{1}{2}x} \cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2 e^{\frac{1}{2}x} \sin\left(\tfrac{\sqrt{3}}{2}x\right)$$

Step 2: Use initial conditions

$y(0) = 1 \Rightarrow$ let $x=0$, $y=1$ $\Rightarrow$

$$1 = c_1 \cdot \cancelto{1}{e^0} \cdot \cancelto{1}{\cos 0} + \cancel{c_2 e^0 \cancelto{0}{\sin 0}}$$

$$\Rightarrow \quad c_1 = 1$$

Note $y'(x) = \frac{1}{2} c_1 e^{\frac{1}{2}x} \cos\left(\tfrac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2} c_1 e^{\frac{1}{2}x} \sin\left(\tfrac{\sqrt{3}}{2}x\right)$

$$+ \frac{1}{2} c_2 e^{\frac{1}{2}x} \sin\left(\tfrac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2} c_2 e^{\frac{1}{2}x} \cos\left(\tfrac{\sqrt{3}}{2}x\right)$$

$y'(0)=3 \Rightarrow$ let $x=0$, $y'=3$

$$\Rightarrow 3 = \frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2$$

$$\Rightarrow \sqrt{3}C_2 = 6 - C_1 = 5$$

$$\Rightarrow C_2 = \frac{5}{\sqrt{3}} = \frac{5}{3}\sqrt{3}$$

Hence the soln to the I.V.P is

$$y(x) = e^{\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{5}{3}\sqrt{3}\,e^{\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

Now let's look at problems # 1, 2, 3, 4

#1: Solve the I.V.P

$$\frac{dy}{dx} = y^2 - 1 \qquad \text{D.E}$$

$$y(0) = 3. \qquad \text{initial condition}$$

Remark: This is a seperable D.E !

$$\frac{dy}{dx} = \underbrace{1}_{f(x)} \cdot \underbrace{(y^2-1)}_{g(y)}$$

First we solve the D.E.

Step 1: check whether $y^2-1=0$ ($g(y)=0$) gives a soln to the D.E.

Note $y^2-1=0 \Rightarrow y=1$ or $y=-1$.

Both of them satisfy

$$\frac{dy}{dx}=y^2-1. \Rightarrow$$ They are solns: $y=1,\ y=-1$

Step 2. Separate the variables and integrate.

$$\Rightarrow \frac{dy}{y^2-1}=dx$$

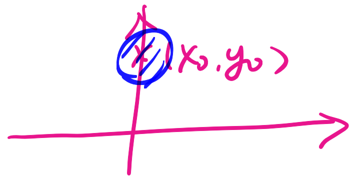
$$\Rightarrow \int\frac{dy}{y^2-1}=\int dx$$

Hint: $\frac{1}{y^2-1}=\frac{1}{2}\frac{1}{y-1}-\frac{1}{2}\frac{1}{y+1}$

$$\Rightarrow \frac{1}{2}\int\frac{dy}{y-1}-\frac{1}{2}\int\frac{dy}{y+1}=\int 1\cdot dx$$

$$\Rightarrow \frac{1}{2}\ln|y-1|-\frac{1}{2}\ln|y+1|=x+C$$

$(x_0, y_0) = (0, 3)$



Recall $y(0) = 3$

We can assume (x, y) are close to $(0, 3)$

Since y is close to 3, $y - 1 > 0$, $y + 1 > 0$

The above becomes

$$\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = x + C$$

Summerize step 1, 2, we have 3 kinds of solns:

$$y = 1 \qquad ①$$

$$y = -1 \qquad ②$$

$$\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = x + C \qquad ③$$

Step 2: Use the initial condition

$$y(0) = 3 \qquad (*)$$

Note ①, ② can not satisfy $(*)$

only ③ can satisfy $(*)$

$\ln 4 = \ln 2^2$
$= 2\ln 2$

Let $x=0$, $y=3$ in ③

$$\Rightarrow \quad \frac{1}{2}\ln(2) - \frac{1}{2}\ln 4 = 0 + C$$

$$\Rightarrow \quad C = \frac{\ln 2}{2} - \frac{\ln 4}{2} = -\frac{\ln 2}{2}$$

Hence the soln to the I.V.P is

$$\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = x - \frac{\ln 2}{2}$$

A common asked Q:

Q: What are explicit soln and implicit soln?

A: Explicit solution means you need to write the solution as

$$"\ y = \phi(x)\ "$$

E.g $y = C_1 e^{-2x} + C_2 x e^{-2x}$

Implicit solution means you can write the solution as an equation that involves x, y.

E.g $\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = x - \frac{\ln 2}{2}$

#2. Find an explicit solution to the D.E.

$$t\frac{dy}{dt} - y = t^2 e^t \text{ for } t>0 \qquad (1)$$

A: Step 0: Make the eqn (1) in to the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

(Here it should be $\frac{dy}{dt} + P(t)y = Q(t)$)

Divide by $t \Rightarrow$

$$\frac{dy}{dt} \underbrace{- \frac{1}{t}}_{P(t)} y = te^t \qquad (2)$$

Step 1: Calculate $\mu(t) = e^{\int P(t)dt}$

$$\Rightarrow \mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln|t| + C}$$

Choose $C=0$ $\rightarrow$ $$= e^{-\ln t} = \frac{1}{t}$$

Step 2: Multiply both sides of (2) by $\mu(t)$

$$\Rightarrow \quad \frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y = e^t$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{t}y\right) = e^t$$

Step 3: Integrate

$$\int \frac{d}{dt}\left(\frac{1}{t}y\right)dt = \int e^t\,dt$$

$$\Rightarrow \quad \frac{1}{t}y = e^t + C$$

$$\Rightarrow \quad y = te^t + Ct$$

explicit sol/n

#3. Solve the I.V.P

$$(3x^2y^2+2xy)dx+(2x^3y+x^2+1)dy=0 \quad \text{D.E}$$

$$y(1)=1 \quad \text{initial condition}$$

Step 0: Reduce to the form $Mdx+Ndy=0$.

already in the right form:

$$\begin{cases} M = 3x^2y^2+2xy \\ N = 2x^3y+x^2+1 \end{cases}$$

Step 1: Check the exactness

$$\begin{cases} \dfrac{\partial M}{\partial y} = 6x^2y+2x \\ \dfrac{\partial N}{\partial x} = 6x^2y+2x \end{cases}$$

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ exact!

Step 2: Find F s.t

$$\begin{cases} \dfrac{\partial F}{\partial x} = M = 3x^2y^2 + 2xy & (1) \\ \dfrac{\partial F}{\partial y} = N = 2x^3y + x^2 + 1 & (2) \end{cases}$$

Integrate (1) w.r.t x $\Rightarrow$ ← regard y as a constant

$$F = \int \frac{\partial F}{\partial x}\,dx = \int (3x^2y^2 + 2xy)\,dx$$

$$= x^3y^2 + x^2y + \boxed{g(y)}$$

constant term

Substitute the above into (2)

$$\Rightarrow \quad \cancel{2x^3y} + \cancel{x^2} + g' = \frac{\partial F}{\partial y} = \cancel{2x^3y} + \cancel{x^2} + 1$$

$$\Rightarrow \quad g'(y) = 1$$

$$\Rightarrow \quad g(y) = \int 1\,dy = y + C$$

choose $C = 0$ $\Rightarrow$ $g(y) = y$

Hence

$$F = x^3y^2 + x^2y + y$$

The soln to the D.E is $(F = c)$

$$x^3y^2 + x^2y + y = c \qquad (3)$$

Step 3: Use the initial condition.

$y(1) = 1 \Rightarrow$ let $x=1$, $y=1$ in (3)

$$1 + 1 + 1 = c$$

$$\Rightarrow c = 3$$

Hence the solution to the I.V.P

$$x^3y^2 + x^2y + y = 3$$

#4. Solve the I.V.P

$$\left(\frac{x^3}{2y} + ye^x\right)dx + (1+2e^x)dy = 0 \qquad (4)$$

$$y(0) = 1$$

Step 0. Make it into the form $Mdx + Ndy = 0$.

already in the right form.

Note
$$\begin{cases} M = \frac{x^3}{2y} + ye^x \\ N = 1 + 2e^x \end{cases}$$

Step 1. Check the exactness

$$\begin{cases} \frac{\partial M}{\partial y} = -\frac{x^3}{2y^2} + e^x \\ \frac{\partial N}{\partial x} = 2e^x \end{cases}$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{not exact!}$$

$\partial_y M = \frac{\partial M}{\partial y}$

$\partial_x N = \frac{\partial N}{\partial x}$

Step 2. Check $\frac{1}{N}(\partial_y M - \partial_x N)$ and $\frac{1}{M}(\partial_x N - \partial_y M)$

(I) If $\frac{1}{N}(\partial_y M - \partial_x N)$ depends only on x, then use the integrating factor

$$\mu(x) = e^{\int \frac{1}{N}(\partial_y M - \partial_x N)\,dx}$$

(II) If $\frac{1}{M}(\partial_x N - \partial_y M)$ depends only on y then use the integrating factor

$$\mu(y) = e^{\int \frac{1}{M}(\partial_x N - \partial_y M)\,dy}$$

—— See Lecture 5, §2.5

Check: (I) $\frac{1}{N}(\partial_y M - \partial_x N)$

$$= \frac{1}{1+2e^x}\left(-\frac{x^3}{2y^2} + e^x - 2e^x\right)$$

Does it only depend on x? No!

(II) check $\frac{1}{M}(\partial_x N - \partial_y M)$

$$= \frac{1}{\frac{x^3}{2y} + y e^x}\left(2e^x - \left(-\frac{x^3}{2y^2} + e^x\right)\right)$$

$$= \frac{1}{y\left(\cancel{\frac{x^3}{2y^2} + e^x}\right)}\left(\cancel{e^x + \frac{x^3}{2y^2}}\right)$$

$$= \frac{1}{y}$$

Does it only depend on y? Yes!

$\Rightarrow$ We can use the integrating factor

$$\mu(y) = e^{\int \frac{1}{y}\, dy} = e^{\ln|y|} = |y|$$

Since the initial condition $y(0)=1$,

(x, y) is close to $(0, 1)$

We can assume y is close to 1,

and therefore $y > 0$.

Hence $\mu = y$.

Step 3. Multiply (4) by μ.

$$\Rightarrow \left(\frac{x^3}{2} + y^2 e^x\right) dx + (y + 2e^x y)\, dy = 0$$

This new eqn is exact.

Step 4: Find F s.t

$$\begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{x^3}{2} + y^2 e^x & (1) \\[2ex] \dfrac{\partial F}{\partial y} = y + 2e^x y & (2) \end{cases}$$

Integrate (1) w.r.t x. $\Rightarrow$

$$F = \int \frac{\partial F}{\partial x}\,dx = \int \left(\frac{x^3}{2} + y^2 e^x\right) dx$$

$$= \frac{1}{8}x^4 + y^2 e^x + \boxed{g(y)}$$

constant term

plug into ② $\Rightarrow$

$$2y e^x + g' = \frac{\partial F}{\partial y} = y + 2e^x y$$

$$\Rightarrow g' = y \quad \Rightarrow g = \int y\,dy = \frac{1}{2}y^2$$

$$\Rightarrow F = \frac{1}{8}x^4 + y^2 e^x + \frac{1}{2}y^2$$

Hence the soln to the D.E is $(F = C)$

$$\frac{1}{8}x^4 + y^2 e^x + \frac{1}{2}y^2 = C \qquad (5)$$

Step 4 Use the initial condition $y(0)=1$

Let $x=0, y=1$ in (5) $\Rightarrow$

$$0 + 1 \cdot e^0 + \frac{1}{2}\cdot 1^2 = C \Rightarrow C = \frac{3}{2}$$

Hence the soln to the I.V.P:

$$\frac{1}{8}x^4 + y^2 e^x + \frac{1}{2}y^2 = \frac{3}{2}$$